ISRAEL JOURNAL OF MATHEMATICS **153** (2006), 45–60

LOW DIMENSIONAL SECTIONS VERSUS PROJECTIONS OF CONVEX BODIES

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ABSTRACT

The structure of low dimensional sections and projections of symmetric convex bodies is studied. For a symmetric convex body $B \subset \mathbb{R}^n$, inequalities between the smallest diameter of rank ℓ projections of B and the largest in-radius of m-dimensional sections of B are established, for a wide range of sub-proportional dimensions. As an application it is shown that every body B in (isomorphic) ℓ-position admits a well-bounded $(\sqrt{n}, 1)$-mixing operator.

1. Introduction

The study of radii of inscribed and circumscribed Euclidean balls on projections (and, dually, sections) of convex bodies in $\mathbb{R}^n$ has been of interest in the asymptotic theory of normed spaces for a long time. This is in part motivated by an extensive investigation of geometric distances of sections and projections of convex bodies in $\mathbb{R}^n$ to the Euclidean ball. In turn, this problem splits in a natural

* Research of this author was partially supported by KBN Grant no. 1 P03A 015 27.

** This author holds the Canada Research Chair in Geometric Analysis.

Received October 14, 2004

way into two separate inequalities between the norms determined by the bodies under consideration. The prime examples of this direction include results on Euclidean sections of convex bodies such as Dvoretzky's theorem, volume ratio theorem and quotient of a subspace theorem; and one-sided estimates such as decay of diameters of random projections and results on diameters of random sections. We refer the reader to [MiS], [P] and [Mi] for more details, and to [GMT] for the latest development in the latter problem. Most techniques used in this context lead to results of a probabilistic nature and estimates for the radii that hold for "generic" (or "random") sections or projections, rather than optimal ones.

The main results of the present paper are additionally motivated by questions concerning the structure of low dimensional (sub-proportional) sections and projections of symmetric convex bodies. We begin with a deterministic approach and we study, for an arbitrary symmetric convex body $B \subset \mathbb{R}^n$, the ratio between the smallest diameter of rank $n-m$ projections of B and the largest in-radius of m-dimensional sections of B. It turns out that the unit ball of ℓ_1^n represents a model situation in this case. It can be easily checked that the discussed ratio for B_1^n is bounded by $2\sqrt{m}$. One of our results, Theorem 3.1, states that for an arbitrary body $B \subset \mathbb{R}^n$ a similar estimate holds (for m in a certain restricted range).

Afterwards we turn our attention to symmetric convex bodies in what we call an isomorphic ℓ-position. In this case the model situations are those of the unit balls B_1^n and B_∞^n. Namely, in Theorem 4.1, we prove that for an arbitrary body in $\mathbb{R}^n$ in isomorphic ℓ-position and for a wide range of dimensions (of projections and sections), an estimate for the ratio between discussed diameters and radii is similar either to that for B_1^n or to that for B_∞^n.

Finally, recall that the problem of existence in every n-dimensional normed space of a "well-complemented" m-dimensional subspace (for a relatively large m) is still open for $m \leq \sqrt{n}$. For $m \geq c\sqrt{n \log n}$ this question was solved in the negative in [G]. In the language used nowadays, this has been done by providing lower estimates for norms of so-called "mixing" operators (see (5.1) for the definition). For more detailed information on this subject see [MT1]. We apply our results on radii for bodies in isomorphic ℓ-position to show (in Theorem 5.1) that every such body $B \subset \mathbb{R}^n$ admits a well-bounded $(\sqrt{n}, 1)$-mixing operator (at least when B is K-convex). This in particular shows that the question on an existence of an $\sqrt{n}$-dimensional well-complemented subspace (of, say, a K-convex Banach space) cannot be solved in the negative by proving a general statement on mixing operators. We conclude the paper by a more detailed discussion of this issue.

ACKNOWLEDGEMENT: The research on this paper was done during a visit of the first-named author at the Department of Mathematical and Statistical Sciences of the University of Alberta. He would like to thank the Department for its support and hospitality.

2. Preliminaries

We shall work with symmetric convex bodies in $\mathbb{R}^n$. We shall use the standard notation in the asymptotic theory of Banach spaces as for example in [P], [T] and [MiS]. In particular, the n-dimensional real Euclidean space is denoted by $(\mathbb{R}^n, \|\cdot\|_2)$ and its unit ball is denoted by B_2^n. For a linear subspace $E \subset \mathbb{R}^n$, the orthogonal projection onto E is denoted by P_E. By a symmetric convex body B in $\mathbb{R}^n$ we always mean a *centrally* symmetric convex body and by $\|\cdot\|_B$ we denote the corresponding norm on $\mathbb{R}^n$. We will often identify such a body B with the normed space $(\mathbb{R}^n, \|\cdot\|_B)$. In particular, for two symmetric convex bodies $B, K \subset \mathbb{R}^n$ and a linear operator $T\colon \mathbb{R}^n \to \mathbb{R}^n$, by $\|T\colon B \to K\|$ we denote the norm of T as operator acting from $(\mathbb{R}^n, \|\cdot\|_B)$ to $(\mathbb{R}^n, \|\cdot\|_K)$. For a symmetric convex body $B \subset \mathbb{R}^n$, we let $B^0 = \{x \mid |(x,y)| \leq 1 \text{ for all } y \in B\}$ denote the polar of B.

Let $B \subset \mathbb{R}^n$ be a symmetric convex body. Recall that

$$M_B^* = \int_{S_{n-1}} \|x\|_{B^0} d\mu(x), \quad M_B = \int_{S_{n-1}} \|x\|_B d\mu(x),$$

where $\mu(\cdot)$ stands for the normalized Haar measure on the unit sphere S_{n-1} in $\mathbb{R}^n$.

An important parameter connected with Dvoretzky's theorem is the transition dimension $k^*(B)$ (we refer the reader to Milman's paper [M-congress] for more information and related results). One defines $k^*(B)$ as the largest dimension k such that the set

$$\mathcal{A}_k = \{H \in G_{n,k} \mid (1/2) M_B^* P_H(B_2^N) \subset P_H(B) \subset 2M_B^* P_H(B_2^N)\} \tag{2.1}$$

has measure

$$\mu_{n,k}(\mathcal{A}_k) \geq 1 - e^{-k}.$$

Here $G_{n,k}$ denotes the Grassmann manifold of k-dimensional subspaces of $\mathbb{R}^n$ and $\mu_{n,k}$ is the normalized Haar measure on $G_{n,k}$. Finally, we set $k(B) =$

$k^*(B^0)$. We have, by results from [Mi-1] and [MiS1],

$$c'(M_B^*/a)^2 n \le k^*(B) \le C'(M_B^*/a)^2 n, \tag{2.2}$$

where $a > 0$ is the smallest number such that $K \subset aB_2^n$, and $C' \ge c' > 0$ are numerical constants, (see also [Families] (1.1), (1.2), (1.3) for the measure concentration used here.)

The following lemma is well known.

LEMMA 2.1: *Let $B \subset \mathbb{R}^n$ be a symmetric convex body and let $F \subset \mathbb{R}^n$ be a k-dimensional subspace, for $1 \le k \le n$. Then*

$$M(B \cap F) \le \sqrt{\pi/2}\sqrt{n/k}M(B) \tag{2.3}$$

and

$$M^*(P_F B) \le \sqrt{\pi/2}\sqrt{n/k}M^*(B). \tag{2.4}$$

Proof: By well-known properties of spherical and Gaussian integrals, for every $n \ge 1$ there is c_n such that

$$\begin{aligned}\int_{S_{n-1}} \|x\| d\mu(x) &= c_n n^{-1/2}(2\pi)^{-n/2}\int_{\mathbb{R}^n} \|x\| \exp(-\|x\|_2^2/2)dx\\ &= c_n n^{-1/2}\int_{\mathbb{R}^n} \|x\| d\gamma_n(x),\end{aligned}$$

for an arbitrary norm $\|\cdot\|$ on $\mathbb{R}^n$, where γ_n is the standard Gaussian probability measure on $\mathbb{R}^n$. It can be easily checked that $c_n \downarrow 1$ when $n \to \infty$, and that $c_1 = \sqrt{\pi/2}$.

Thus we have

$$\begin{aligned}M(B \cap F) &= \int_{S_{n-1}\cap F} \|x\|_{B\cap F} d\mu_F(x)\\ &= c_k k^{-1/2}\int_F \|x\|_B d\gamma_k(x)\\ &\le c_k k^{-1/2}\int_{\mathbb{R}^n} \|x\|_B d\gamma_n(x)\\ &= (c_k/c_n)(n/k)^{1/2}M(B)\\ &\le \sqrt{\pi/2}(n/k)^{1/2}M(B),\end{aligned}$$

which proves (2.3).

Now (2.4) follows from (2.3) by duality. ∎

The notion of the ℓ-ellipsoid and the ℓ-position of a symmetric convex body, introduced by Figiel and Tomczak–Jaegermann (cf. e.g. [T], [P]), plays an important role in the study of linear structure of normed spaces. It is known that the ℓ-ellipsoid is unique and hence an ℓ-position of a given symmetric convex body is unique up to a rotation. Here it is sufficient to use an isomorphic variant of these notions. So for a symmetric convex body $B \subset \mathbb{R}^n$ and $C_0 \geq 1$ we say that B is in C_0-isomorphic ℓ-position whenever

$$M_B = M_B^* \leq \sqrt{C_0 \kappa(B)}, \tag{2.5}$$

where $\kappa(B)$ is the K-convexity constant of B (cf. [T], [P]). (Let us note that if a body K is in the ℓ-position and B satisfies $(1/\sqrt{C_0})K \subset B \subset \sqrt{C_0}K$, then B is indeed in C_0-isomorphic ℓ-position. Although the converse implication is not necessarily true, the property (2.5) is in fact sufficient for our purposes.)

It can be easily verified that if a symmetric convex body $B \subset \mathbb{R}^n$ is in C_0-isomorphic ℓ-position, then

$$\frac{1}{\sqrt{C_0 n \kappa(B)}} B_2^n \subset B \subset \sqrt{C_0 n \kappa(B)} B_2^n. \tag{2.6}$$

Let us also recall that $\kappa(B) \leq C \log n$, for any symmetric convex body $B \subset \mathbb{R}^n$, where C is a suitable numerical constant (cf. e.g., [T], [P]).

In this paper we are interested in comparing maximal radii of Euclidean balls inscribed into k-dimensional sections of a body B with minimal radii of Euclidean balls circumscribed on m-dimensional orthogonal projections of the body. We introduce appropriate notions.

Let $B \subset \mathbb{R}^n$ be a symmetric convex body. For any $1 \leq k \leq n$ we let

$$\rho_k(B) = \max_E \max\{r \mid rB_2^n \cap E \subset B \cap E\}, \tag{2.7}$$

where the first maximum is taken over all k-dimensional subspaces E of $\mathbb{R}^n$.

Similarly, we define the dual notion. For any $1 \leq k \leq n$ we let

$$\alpha_k(B) = \min_E \min\{R \mid P_E B \subset R P_E B_2^n\}, \tag{2.8}$$

where the first minimum is taken over all k-dimensional subspaces E of $\mathbb{R}^n$. In the sequel, for K of the form $B \cap E$ or $P_E B$, we shall write $\alpha(K)$ instead of $\alpha_{\dim E}(K)$, and $\rho(K)$ instead of $\rho_{\dim E}(K)$.

Clearly, $\rho_k(B^0) = \alpha_k(B)$ for all $B \subset \mathbb{R}^n$ and $1 \leq k \leq n$.

To put these notions in a perspective, recall that for an arbitrary symmetric convex body $B \subset \mathbb{R}^n$ the asymptotic behavior of radii of Euclidean balls circumscribed on "random" k-dimensional projections of B is well known. Namely, a so-called "standard shrinking" principle (which is in fact a one-sided estimate in Milman's randomized version of Dvoretzky's theorem; cf., e.g., [MiS]) states that there exists an absolute constant $C' > 0$ such that

$$\mu_{n,k}\{H \in G_{n,k} \mid \alpha(P_H B) \le C'\alpha(B)\sqrt{k/n}\} \ge 1 - e^{-k}, \tag{2.9}$$

for $k \ge k^*(B)$. We refer the reader to [ST], Lemma 3.4 for a more precise statement of (2.9) and its proof.

3. Sections and projections for bodies in general position

In this section we consider n-dimensional symmetric convex bodies satisfying only very mild restrictions on their position. We discuss the relation between maximal radii of Euclidean balls inscribed into sections of the body and minimal diameters of its projections. The next theorem shows that either a body admits an $(n-k)$-dimensional orthogonal projection with a small diameter, or it has an m-dimensional section containing a relatively large Euclidean ball (where $1 \le k \le n$ and m is of order $k/\log n$). Observe that the inequality in the theorem below is optimal, up to constant 4, for the unit ball B_1^n of l_1^n.

THEOREM 3.1: *Let $B \subset \mathbb{R}^n$ be a symmetric convex body and let $p > 0$ be such that $n^{-p}B_2^n \subset B \subset n^p B_2^n$. Then for every $1 \le k \le n$ we have*

$$\alpha_{n-k}(B) \le 4\sqrt{m}\rho_m(B), \tag{3.1}$$

for any $m \le \lceil k/(2^{13}p\log n)\rceil$.

Remark: In terms of Gelfand numbers of operators inequality (3.1) states that

$$c_k(Id\colon l_2^n \to (\mathbb{R}^n, B^0))\ c_{n-m}(Id\colon l_2^n \to (\mathbb{R}^n, B)) \le 4\sqrt{m},$$

where B and k and m are as in Theorem 3.1. It was pointed out to the authors by V. D. Milman that a result similar in spirit has been proved in [Mi0] where an upper estimate for the product

$$c_k(Id\colon (\mathbb{R}^n, B) \to l_2^n)\ c_\ell(Id\colon (\mathbb{R}^n, B^0) \to l_2^n)$$

was established for suitable $k + \ell > n$, without additional assumptions on the body B.

Proof: Fix $1 \le k < n$. We shall prove that there exists an m-dimensional section of B which contains the Euclidean ball of radius $\alpha_{n-k}(B)/(4\sqrt{m})$, for some m satisfying $m \ge \lceil ck/\log n\rceil$, where $c > 0$ depends on p only.

We shall define an auxiliary operator $T\colon \mathbb{R}^n \to \mathbb{R}^n$ (depending on B and k) by the following general procedure.

Let $F_1 = \mathbb{R}^n$. Set $a_1 = \alpha(B)$ and pick any $x_1 \in B$ satisfying $\|x_1\|_2 = a_1$. Let $F_2 = \mathrm{span}[x_1]^\perp$, $a_2 = \alpha(P_{F_2}B)$ and pick $x_2 \in P_{F_2}B$ satisfying $\|x_2\|_2 = a_2$. Repeat the inductive procedure $k-2$ more times to obtain $x_1, x_2, \ldots, x_k \in \mathbb{R}^n$ and $\mathbb{R}^n = F_1 \supset F_2 \supset \cdots \supset F_k \supset F_{k+1}$, with $\dim F_i = n - i + 1$, and $a_1 \geq a_2 \geq \cdots \geq a_k$ satisfying $x_i \in P_{F_i}B$, $\|x_i\|_2 = a_i = \alpha(P_{F_i}B)$ and $F_{i+1} = \mathrm{span}[x_1, \ldots, x_i]^\perp$, for $i = 1, 2, \ldots, k$.

Set $u_i = x_i/a_i$, for $i = 1, 2, \ldots, k$. Clearly $\{u_i\}_{i=1}^k$ form an orthonormal system and $F := F_{k+1}$ is the orthogonal complement of its span. Define $T\colon \mathbb{R}^n \to \mathbb{R}^n$ by

$$T|_F = Id_F \quad \text{and} \quad Tu_i = (a_k/a_i)u_i \quad \text{for } i = 1, 2, \ldots, k. \tag{3.2}$$

Observe that T commutes with all P_{F_i}'s.

First we are going to investigate the norm $\|T : B \to B_2^n\|$.

Lemma 3.2: *With the above notation we have*

$$\|T\colon B \to B_2^n\| \leq a_k\sqrt{2(1 + \log(a_1/a_k))}.$$

Proof: Fix an arbitrary $z \in B$ and write it in the form $z = \sum_{i=1}^k t_i u_i + P_F z$. We have

$$\|Tz\|_2^2 = \|P_F Tz\|_2^2 + \|P_{F^\perp} Tz\|_2^2. \tag{3.3}$$

Clearly, by the definition of T,

$$\|P_F Tz\|_2 = \|TP_F z\|_2 = \|P_F z\|_2 \leq a_k,$$

and

$$\|P_{F^\perp} Tz\|_2 = \|TP_{F^\perp} z\|_2 = \left(\sum_{i=1}^k t_i^2 (a_k/a_i)^2\right)^{1/2}.$$

By the definition of x_i's, every $z = \sum_{i=1}^k t_i u_i + P_F z \in B$ satisfies, for $j = 1, \ldots, k$,

$$\sum_{i=j}^k t_i^2 = \|P_{F^\perp} P_{F_j} z\|_2^2 \leq \alpha(P_{F_j}B)^2 = a_j^2.$$

Therefore we get

$$\max_{z \in B} \|Tz\|_2^2 \leq a_k^2 + \max_w \sum_{i=1}^k t_i^2 (a_k/a_i)^2, \tag{3.4}$$

where the latter maximum is taken over all vectors w of the form $w = \sum_{i=1}^k t_i u_i$ satisfying the constraints inequalities

$$\sum_{i=j}^{k} t_i^2 \le a_j^2 \quad \text{for all } j = 1, \dots, k. \tag{3.5}$$

To consider the latter maximum in (3.4) we observe that the substitution $s_j = t_j^2$ $(j = 1, \dots, k)$ reduces it to a simple linear programming problem of maximizing the function $\phi(s) := \sum_{i=1}^k s_j (a_k/a_i)^2$ over $s = (s_1, \dots, s_k) \in \mathbb{R}^k$ satisfying $\sum_{i=j}^k s_i \le a_j^2$ for $j = 1, \dots, k$; and recall additionally that $a_1 \ge \cdots \ge a_k > 0$.

Since in the formula for ϕ the coefficient by s_k is the largest, one can show that the maximum of ϕ is attained when s_k is the largest possible, that is, $s_k = a_k^2$. Repeating the same argument again k times yields that ϕ attains its maximum for $s_k = a_k^2$ and $s_j = a_j^2 - a_{j+1}^2$ for $j = 1, \dots, k-1$. One can check directly from the form of ϕ that the same remains true if some of the a_j's are equal, although in this case a point where the maximum is attained is not unique. Thus the latter maximum in (3.4) is equal to

$$a_k^2 + \sum_{i=1}^{k-1} (a_k/a_i)^2 (a_i^2 - a_{i+1}^2) = a_k^2 \left(k - \sum_{i=1}^{k-1} (a_{i+1}/a_i)^2 \right). \tag{3.6}$$

Observe that

$$\prod_{i=1}^{k-1} (a_{i+1}/a_i)^2 = (a_k/a_1)^2.$$

Thus $\sum_{i=1}^{k-1} (a_{i+1}/a_i)^2$ is the smallest when all terms are equal and thus are equal to $(a_k/a_1)^{2/(k-1)}$. Therefore, by (3.4),

$$\max_{z \in B} \|Tz\|_2^2 \le a_k^2 ((k+1) - (k-1)(a_k/a_1)^{2/(k-1)}). \tag{3.7}$$

To estimate the right hand side expression set $t = 1/(k-1)$ and $c = (a_1/a_k)^2 \ge 1$, and note that $(k-1) - (k-1)(a_k/a_1)^{2/(k-1)} = c^{-t}(c^t - 1)/t$. By the mean value theorem applied to the function $f(t) = c^t$ we get $(c^t - 1)/t \le c^t \log c$. Hence we get

$$\max_{z \in B} \|Tz\|_2^2 \le a_k^2 (2 + 2\log(a_1/a_k)) = 2a_k^2 (1 + \log(a_1/a_k)),$$

which completes the proof of the lemma. ∎

Now, returning to the proof of the theorem, for $i = 1, \dots, k$, pick $y_i \in B$ such that $P_{F_i} y_i = x_i$, and set $z_i = Ty_i$. By the lemma, for $i = 1, \dots, k$ we have

$$\|z_i\|_2 \leq a_k \sqrt{2(1 + \log(a_1/a_k))}.$$

On the other hand, it follows from (3.2) that

$$(z_i, u_i) = (y_i, T^* u_i) = (a_k/a_i)(y_i, u_i) = (a_k/a_i)(x_i, u_i) = a_k,$$

for $i = 1, \dots, k$. Using a result of Bourgain–Tzafriri ([BT], Theorem 7.1) in the form presented in [BS], Lemma B, we get that there exists a subset $\sigma \subset \{1, \dots, k\}$ with cardinality $m := |\sigma| \geq 2^{-11}(1 + \log(a_1/a_k))^{-1} k$ such that for every $t_1, \dots, t_k \in \mathbb{R}$ we have

$$\Big\| \sum_{i \in \sigma} t_i z_i \Big\|_2 \geq (a_k/4) \Big(\sum_{i \in \sigma} t_i^2 \Big)^{1/2}.$$

Set $E_0 = \operatorname{span}[z_i]_{i \in \sigma}$. It is well known and easy to verify that for the operator $S: \mathbb{R}^k \to E_0$ defined by $Se_i = z_i$, for $i \in \sigma$, and $Se_i = 0$ otherwise, the estimate above is equivalent to $S(B_2^k) \supset (a_k/4) B_2^n \cap E_0$. This in turn easily implies that

$$T(B) \cap E_0 \supset \operatorname{conv}[Ty_i]_{i \in \sigma} = \operatorname{conv}[z_i]_{i \in \sigma} \supset (a_k/4\sqrt{m}) B_2^n \cap E_0. \tag{3.8}$$

Also note that for any $m' \leq \tilde{m}$, and considering $\sigma' \subset \sigma$ with $|\sigma'| = m'$ we get a subspace $E' \subset E_0$ with $\dim E' = m'$ and such that

$$T(B) \cap E' \supset (a_k/4\sqrt{m'}) B_2^n \cap E'. \tag{3.9}$$

Set $m' := \lceil 2^{-13} k/(p \log n) \rceil$. Since $n^{-p} \leq a_k \leq a_1 \leq n^p$, then clearly $m' \leq m$. Fix any $\sigma' \subset \sigma$ with $|\sigma'| = m'$. Since T is a contraction with respect to the Euclidean norm, setting $E = \operatorname{span}[y_i]_{i \in \sigma'}$, we get $\dim E = m'$ and

$$B \cap E \supset \operatorname{conv}[y_i]_{i \in \sigma'} \supset (a_k/4\sqrt{m'}) B_2^n \cap E. \tag{3.10}$$

By the definitions of $\rho_{m'}(B)$ and $\alpha_k(B)$, (3.10) yields

$$\rho_{m'}(B) \geq (a_k/4\sqrt{m'}) \geq (\alpha_{n-k}(B)/4\sqrt{m'}),$$

which completes the proof. ∎

Remark: For future reference, observe that the proof above in fact shows that if a symmetric convex body $B \subset \mathbb{R}^n$ satisfies $rB_2^n \subset B \subset RB_2^n$ for some $R \geq r > 0$, then for every $1 \leq k \leq n$ the inequality (3.1) holds for any $m \leq \lceil 2^{-11}k/(1 + \log R - \log r) \rceil$.

As a byproduct of the proof of Theorem 3.1, we get the following proposition which may be of independent interest. In particular, note a very weak dependence of the cardinality of σ on the maximum of the norms of vectors. We leave the proof to the reader.

PROPOSITION 3.3: *Let $1 \leq k \leq n$ and let $z_1, \ldots, z_k \in \mathbb{R}^n$ satisfy*

$$\max_{1 \leq i \leq k} \|z_i\|_2 = a > 0 \quad \text{and} \quad \min_{1 \leq i \leq k} \operatorname{dist}(z_i, \operatorname{span}\{z_j \mid j \neq i\}) = b > 0.$$

Then there exists a subset $\sigma \subset \{1, \ldots, k\}$ with $|\sigma| \geq 2^{-11}k/\log(1 + a/b)$ such that

$$\operatorname{conv}\{\pm z_i \mid i \in \sigma\} \supset \frac{b}{4\sqrt{|\sigma|}} B_2^n \cap \operatorname{span}\{z_i \mid i \in \sigma\}.$$

4. Sections and projections for bodies in ℓ-position

For symmetric convex bodies in isomorphic ℓ-positions it is possible to extend Theorem 3.1 to a full range of dimensions ℓ of projections, for $m \leq c_1 n/\log n$ and $m \leq \ell \leq n - c_2 m \log n$ where $c_1, c_2 > 0$ are numerical constants. Namely, we have

THEOREM 4.1: *There exists a constant $C \geq 1$ such that for an arbitrary symmetric convex body $B \subset \mathbb{R}^n$ in a C_0-isomorphic ℓ-position, every $1 \leq m \leq n/(2^{14} \log(C_0 n^2))$ and every ℓ satisfying $m \leq \ell \leq n - 2^{13} m \log(C_0 n^2)$ one has*

$$\min\left(\frac{\alpha_\ell(B)}{\rho_m(B)}, \frac{\alpha_m(B)}{\rho_\ell(B)}\right) \leq C_0 C \max\left(\kappa(B), \sqrt{\frac{\ell m}{n}}\right). \tag{4.1}$$

By letting $m = \ell$ we get

COROLLARY 4.2: *For every symmetric convex body $B \subset \mathbb{R}^n$ in a C_0-isomorphic ℓ-position and every $1 \leq m \leq n/(2^{14} \log(C_0 n^2))$ we have*

$$\frac{\alpha_m(B)}{\rho_m(B)} \leq C_0 C \max\left(\kappa(B), \frac{m}{\sqrt{n}}\right),$$

where C is the constant from Theorem 4.1.

In particular, for $m = \ell = \lceil \sqrt{n} \rceil$, we get

$$\alpha_m(B) \leq C_0 C \kappa(B) \rho_m(B).$$

We shall prove Theorem 4.1 in a stronger randomized version. Also, note that for each of the unit balls B_1^n and B_∞^n, both terms in the left hand side of (4.1) are approximately equal and therefore each of these terms satisfies the inequality. In contrast, in the theorem below, B_1^n satisfies condition (i) but not (ii), while B_∞^n satisfies (ii) but not (i).

THEOREM 4.3: *There exists a constant $C \geq 1$ such that for an arbitrary symmetric convex body $B \subset \mathbb{R}^n$ in a C_0-isomorphic ℓ-position we have: For every $1 \leq m \leq n/(2^{14}\log(C_0 n^2))$ there exists a subspace $F \subset \mathbb{R}^n$ with $\dim F \geq n - 2^{13} m \log(C_0 n^2)$ such that for every $m \leq \ell \leq \dim F$ at least one of the conditions is satisfied:*

(i) *The measure $\mu_{\dim F,\ell}$ of the subset of $G_{F,\ell}$ of all $H \subset F$ such that*

$$\alpha(P_H B) \leq C_0 C \max(\kappa(B), \sqrt{\ell m/n})\rho_m(B)$$

is larger than or equal to $1 - 2\exp(-\ell)$.

(ii) *The measure $\mu_{\dim F,\ell}$ of the subset of $G_{F,\ell}$ of all $H \subset F$ such that*

$$\alpha_m(B) \leq C_0 C \max(\kappa(B), \sqrt{\ell m/n})\rho(B \cap H)$$

is larger than or equal to $1 - 2\exp(-\ell)$.

Remark: In fact, the theorem above allows a "further randomization" to random subspaces H' of $\mathbb{R}^n$ instead of random $H \subset F$. For example, with the assumptions of Theorem 4.3, if (i) is satisfied then we also have

$$\mu_{n,\ell}\Big\{H' \in G_{n,\ell} \mid \alpha(P_{H'}P_F B) \leq C_0 C \max\Big(\kappa(B), \sqrt{\frac{\ell m}{n}}\Big)\rho_m(B)\Big\} \geq 1 - 2e^{-\ell}. \tag{4.2}$$

Indeed, since $\ell \leq \dim F$, for $H' \in G_{n,\ell}$ we may write $P_{H'}|F\colon F \to H'$ in the form

$$P_{H'}|F = T_{H'} P_H|F,$$

where $H = (\ker P_{H'}|F)^\perp \cap F = (H')^\perp \cap F$ and $T_{H'} = P_{H'}|H\colon H \to H'$ is a contraction. By the uniqueness of Haar measure on $G_{F,\ell}$, H is uniformly distributed in F (note that the measure of H' such that $\dim h = \ell$ is equal to 1). This easily implies (4.2). The reader can consult [MT2], Proposition 3.2, for a more general argument.

Proof of Theorem 4.3: Let $1 \leq m \leq n/(2^{14}\log(C_0 n^2))$ and set

$$k = \lfloor 2^{12} m \log(C_0 n^2) \rfloor.$$

Note that $n - 2k \geq n/2$. For any symmetric convex body $K \subset \mathbb{R}^n$ in a C_0-isomorphic ℓ-position, combining (2.6) with the remark after the proof of Theorem 3.1 (with, for example, $R = 1/r = \sqrt{C_0 n}$), we get

$$\alpha_{n-k}(K) \leq 4\sqrt{m}\rho_m(K). \tag{4.3}$$

Later on we shall use this inequality for both B and B^0.

Pick $(n-k)$-dimensional subspaces $F_1, F_2 \subset \mathbb{R}^n$ such that

$$\alpha(P_{F_1}B) = \alpha_{n-k}(B) \quad \text{and} \quad \alpha(P_{F_2}B^0) = \alpha_{n-k}(B^0),$$

and let $F = F_1 \cap F_2$. Clearly, $\dim F \geq n - 2k \geq n/2$.

Fix an arbitrary $m \leq \ell \leq n - 2k$.

If $k^*(P_F B) \leq \ell$, then by (2.9) applied to $P_F(B) \subset F$ and ℓ we get

$$\mu_{\dim F, \ell}\{H \in G_{F,\ell} \mid \alpha(P_H P_F B) \leq C'\alpha(P_F B)\sqrt{\ell/\dim F}\} \geq 1 - e^{-\ell}.$$

Note that $P_H P_F = P_H$ and $\alpha(P_F B)\sqrt{\ell/\dim F} \leq \alpha_{n-k}(B)\sqrt{2\ell/n}$. Combining this with (4.3) for $K = B$ shows that (i) holds.

If $k^*(P_F B^0) \leq \ell$, then applying the same argument to B^0 instead of B proves (i) for B^0. By duality this implies that (ii) is satisfied (for B).

Thus it remains to prove the theorem in the case when $k^*(P_F B) \geq \ell$ and $k^*(P_F B^0) \geq \ell$. The first inequality implies, by the definition of $k^*(\cdot)$, that

$$\mu_{\dim F, \ell}\{H' \in G_{F,\ell} \mid \alpha(P_{H'} P_F B) \leq 2M^*(P_F B)\} \geq 1 - e^{-\ell}. \tag{4.4}$$

In a similar way, the second inequality implies, by duality,

$$\mu_{\dim F, \ell}\{H'' \in G_{F,\ell} \mid \rho(B \cap H'') \geq (2M(B \cap F))^{-1}\} \geq 1 - e^{-\ell}. \tag{4.5}$$

Taking $H \subset F$ from the intersection of the two sets considered above we get

$$\alpha(P_H B) \leq 2M^*(P_F B) 2M(B \cap F)\rho(B \cap H).$$

Thus for H in a subset of $G_{F,\ell}$ of measure $\geq 1 - 2\exp(-\ell)$ we have

$$\alpha(P_H B) \leq 4C_0 \kappa(B)\rho(B \cap H).$$

This estimate implies both (i) and (ii). Indeed, since $\rho(B \cap H) \leq \rho_\ell(B) \leq \rho_m(B)$, this implies (i). Similarly, (ii) is satisfied as well (as $\alpha(P_H B) \geq \alpha_m(B)$).

It is noteworthy that in this case a random ℓ-dimensional subspace of $(\mathbb{R}^n, \|\cdot\|_B)$ is a $4C_0\kappa(B)$-complemented Euclidean. ∎

5. Boundedness of mixing operators

Over the years, the notion of mixing operators proved to be a useful tool in the study of linear-geometric properties of finite-dimensional Banach spaces (or equivalently, of symmetric convex bodies in $\mathbb{R}^n$). It was formally introduced in [S2], however its defining property has been already used in earlier papers [G], [S1], [M1]. For more information on the subject the reader may consult [MT1].

We begin by recalling the definition. For $1 \leq m \leq n/2$, we say that an operator $T\colon \mathbb{R}^n \to \mathbb{R}^n$ is $(m,1)$-mixing provided that there exists a subspace $E \subset \mathbb{R}^n$ with $\dim E \geq m$ satisfying

$$\text{dist}(Tx, E) = \|P_{E^\perp} Tx\|_2 \geq \|x\|_2, \tag{5.1}$$

for every $x \in E$.

It is known that for a symmetric convex body $B \subset \mathbb{R}^n$, lower estimates for norms of $(m,1)$-mixing operators $T\colon B \to B$ (for a suitable m) control various linear-geometric invariants of B, and we refer the reader to [MT1] (Sections 5 and 6), and references therein, for many examples by various authors. In particular, there is a simple connection between projections and mixing operators, namely, if P is a rank m (not necessarily orthogonal) projection, with $m \leq n/2$, then $2P$ is $(m,1)$-mixing.

Let $\mathcal{M}$ be one of the following classes of $n \times n^2$ rectangular matrices, endowed with natural probability measures: (i) matrices with independent identically distributed Gaussian entries; (ii) matrices of the form $Q = PU$ where $P\colon \mathbb{R}^{n^2} \to \mathbb{R}^n$ is the restriction to the first n coordinates, and $U \in \mathcal{O}_{n^2}$; and finally, (iii) a matrix with independent columns uniformly distributed on the sphere S_{n-1}. Consider elements of $\mathcal{M}$ as operators $Q\colon \mathbb{R}^{n^2} \to \mathbb{R}^n$. A combination of known (past and recent) estimates can be summarized in

for a "random" $Q \in \mathcal{M}$, the body $B = Q(B_1^{n^2}) \subset \mathbb{R}^n$ has the property:

$$\|T\colon B \to B\| \geq cm/\sqrt{n(1+\log n)}, \tag{5.2}$$

for every $(m,1)$-mixing operator $T\colon \mathbb{R}^n \to \mathbb{R}^n$, for $1 \leq m \leq n/2$, where $c > 0$ is a universal constant.

The main result in this section yields that for an arbitrary n-dimensional symmetric convex body B in a C_0-isomorphic ℓ-position, one cannot essentially improve uniform lower estimates in (5.2), for the norms of all $(m,1)$-mixing operators. In particular, it shows that for any body there always exists a well-bounded $(\lceil c\sqrt{n}\rceil, 1)$-mixing operator.

In fact we get a stronger result: there exist mixing operators which factor through ℓ_2 in a well-bounded way, in other words, their ℓ_2-factorable norm is well-bounded.

Recall that for an operator $T: \mathbb{R}^n \to \mathbb{R}^n$ and a symmetric convex body $B \subset \mathbb{R}^n$, the ℓ_2-factorable norm $\gamma_2(T: B \to B)$ is defined by

$$\gamma_2(T: B \to B) = \inf \|S_1: B \to B_2^n\| \|S_2: B_2^n \to B\|,$$

where the infimum is taken over all operators S_1, S_2 satisfying $T = S_2 S_2$.

THEOREM 5.1: *For every symmetric convex body $B \subset \mathbb{R}^n$ in a C_0-isomorphic ℓ-position, and for every $1 \leq m \leq n/(2^{14}\log(C_0 n^2)$, there exists a $(m,1)$-mixing operator $T: \mathbb{R}^n \to \mathbb{R}^n$ satisfying*

$$\gamma_2(T: B \to B) \leq C_0 C \max(\kappa(B), m/\sqrt{n}),$$

where C is a universal constant. In particular,

$$\|T: B \to B\| \leq C_0 C \max(\kappa(B), m/\sqrt{n}).$$

Proof: Let $B \subset \mathbb{R}^n$ be a symmetric convex body in a C_0-isomorphic ℓ-position. Fix $1 \leq m \leq n/(2^{14}\log(C_0 n^2)$. Pick subspaces $E, H \subset \mathbb{R}^n$ with $\dim E = \dim H = m$ such that

$$\alpha_m(B) = \alpha(P_H B) \quad \text{and} \quad \rho_m(B) = \rho(B \cap E).$$

Let $U: \mathbb{R}^n \to \mathbb{R}^n$ be an arbitrary isometry (with respect to the norm $\|\cdot\|_2$) such that $U(H) = E$, and set $T = 4UP_H$. Clearly,

$$\|P_H: B \to B_2^n\| = \alpha(P_H B),$$

and

$$\|UP_H: B_2^n \to B\| = \|U|_H: B_2^n \cap H \to B \cap E\| = \rho(B \cap E)^{-1}.$$

So, by Corollary 4.2,

$$\gamma_2(T: B \to B) \leq 4C_0 C \max(\kappa(K).m/\sqrt{n}). \tag{5.3}$$

On the other hand, the sequence of s-numbers of T is the same as for the operator $4P_H$, and for this latter operator we have $s_1(4P_H) = \cdots = s_m(4P_H) = 4$ and $s_{m+1}(4P_H) = 0$. A direct calculation shows that this form of s-numbers implies that T is $(m/4, 1)$-mixing (cf. [M2], Lemma 2.6). This concludes the proof. ∎

In fact, there is a randomized construction of $(m, 1)$-mixing operators with similar control of the norm. Namely, let $E \subset \mathbb{R}^n$ be the subspace from the proof of Theorem 5.1, and let $F \subset \mathbb{R}^n$ be as in Theorem 4.3. Assume first that (i) in Theorem 4.3 is satisfied. For an operator U in the orthogonal group $\mathcal{O}_n$ let $T_U = P_E U P_F$. Since for $H' = U^{-1}E$ we have

$$\alpha(P_E T_U B) = \alpha(U^{-1} P_E U P_F B) = \alpha(P_{H'} P_F B),$$

then by the same argument as in (5.3) and by the remark after Theorem 4.3, we infer that the measure on $\mathcal{O}_n$ of the set of all $U \in \mathcal{O}_n$ such that

$$\gamma_2(T_U\colon B \to B) \leq C_0 C \max(\kappa(B).m/\sqrt{n})$$

is larger than or equal to $1 - 2\exp(-\ell)$. To prove the mixing condition, note that by [MT2], Proposition 3.1, for a "random" $U \in \mathcal{O}_n$, the sequence of s-numbers of T_U satisfies that $s_1(T_U) \geq \cdots \geq s_m(T_U) \geq \delta$ and $s_{m+1}(T_U) = 0$, for some numerical constant $\delta > 0$. Such distribution of s-numbers yields, by the same argument as before, that $(4/\delta)T_U$ is $(m/4, 1)$ mixing. We leave the details for the reader.

On the other hand, observe that condition (ii) in Theorem 4.3 is equivalent to condition (i) for B^0 replacing B. Consequently, if (ii) is satisfied then the discussion above applies to B^0, yielding a "randomized" $(m/4, 1)$-mixing operator from B^0 to B^0, with good control of the γ_2-norm. Since the γ_2-norm and mixing properties of operators are preserved when passing to dual operators, this implies the existence of a required operator on B itself.

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